Algebraic Geometry Lecture 13 – Intersections in Projective Space Part I

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Introduction.

Suppose we have two varieties Y and Z in $\mathbb{P}^n$. What can we say about their intersection $Y \cap Z$? One would hope this too was a variety, but alas it is not.

Example Let $Y, Z \in \mathbb{P}^3$ be the quadric surfaces given by

$$Y : x^2 - yw = 0$$
$$Z : xy - zw = 0.$$

Then

$$W = Y \cap Z = \{[x : y : z : w] \in \mathbb{P}^3 \mid x^2 - yw = xy - zw = 0\}.$$

For any $s, t \in \mathbb{R}$ not both zero we have

$$[st^2 : s^2t : s^3 : t^3] \in W.$$

And for any $u, v \in \mathbb{R}$ not both zero we have

$$[0 : u : v : 0] \in W.$$

These are both projective varieties, the first is the twisted cubic given by

$$T : \begin{cases} x^2 - yw = 0 \\ y^2 - xz = 0 \\ xy - zw = 0. \end{cases}$$

And the second is the projective line given by

$$L : \begin{cases} x = 0 \\ w = 0. \end{cases}$$

It is easy to see they are both properly contained in W, i.e. $T, L \subsetneq W$, so $T \cup L \subset W$. To show that $W \subset T \cup L$ we note that this union is given by

$$T \cup L : \begin{cases} x^2 - yw = 0 \quad (1) \quad \text{OR} \\ y^2 - xz = 0 \quad (2) \quad \text{OR} \\ xy - zw = 0 \quad (3) \quad \text{OR} \\ x = 0 \quad (4) \quad \text{OR} \\ w = 0 \quad (5) \end{cases}$$

Conditions (1)-(3) come from T and conditions (4) and (5) come from L. If $P = [x : y : z : w] \in W$ then P satisfies (1) and (3). If it also satisfies (2) then P is in T and we're done. If not then we have

$$y^2 \neq xz.$$

In particular then $y \neq 0$, else from condition (1) we would have $x = 0$ as well and then condition (2) would be satisfied. Since $y \neq 0$ we can multiply condition (3) by y without reducing to $0 = 0$:

$$\begin{aligned} xy &= zw \\ xy^2 &= zwy \qquad \text{(multiply by } y^2\text{)} \\ xy^2 &= zx^2 \qquad \text{(use condition (1))} \\ x(y^2) &= x(xz). \end{aligned}$$

But $y^2 \neq xz$ so we must have $x = 0$. Then $w = 0$ since $x^2 = yw$ and $y \neq 0$. So P satisfies conditions (4) and (5) and hence is in L. Thus

$$W = T \cup L$$

and so W is reducible, hence not a variety.

So the intersection of two varieties need not be another variety. But when it *is* a variety you might hope that the ideals would behave themselves. Specifically you'd want the ideal of the intersection to be the sum of the original ideals. But even that isn't true.

Example Let

$$\begin{aligned} &C : x^2 - yz = 0 \\ &L : y = 0 \end{aligned}$$

be two varieties in $\mathbb{P}^2$. Then

$$C \cap L = \{[x : y : z] \in \mathbb{P}^2 \mid x^2 - yz = y = 0\} = \{[0 : 0 : 1]\}$$

is the variety given by $x = 0, y = 0$. But

$$\begin{aligned} &I(C) = (x^2 - yz) \\ &I(L) = (y) \\ &I(C \cap L) = (x, y). \end{aligned}$$

So while $I(C) + I(L)$ has z dependence, $I(C \cap L)$ doesn't. So

$$I(C) + I(L) \neq I(C \cap L).$$

So what can we say about this intersection? We already know from the proof that the Zariski topology is a topology that the intersection of any countable family of algebraic sets is algebraic. In particular the intersection of two projective varieties is an algebraic set and so we can instead ask about this sets' irreducible components.

What we really want is a generalisation of Bézout's theorem:

Theorem (Bézout)**.** *If Y and Z are plane curves (i.e. in $\mathbb{P}^2$) of degrees d and e respectively and $Y \neq Z$ then $Y \cap Z$ consists of de points counted with multiplicities.*

Generalising this isn't just a case of counting points in higher dimensions. Two planes intersect in one line, so we need to work out what is meant by the degree of a general variety and how to generalise the idea of "multiplicity of intersection". Historically the second part was the hardest and many failed definitions were tried before the correct one to use was found.

Dimension theorems.

Recall that the dimension of a variety V is given by

$$\dim(V) = \operatorname{trdeg}_k k(V).$$

In linear algebra we learn that if U and V are "sufficiently general" subspaces of dimensions r and s of a vector space W of dimension n, then $U \cap V$ is a subspace of dimension $r + s - n$. In algebraic geometry we have the following two results.

Affine Dimension theorem. *Let $Y, Z \subset \mathbb{A}^n$ be varieties of dimensions r and s respectively. Then every irreducible component W of $Y \cap Z$ has dimension $\geqslant r + s - n$.*

Idea of proof. Prove the result when Z is a hypersurface, then reduce the general case to this special one. $\square$

Projective Dimension theorem. *Let $Y, Z \subset \mathbb{P}^n$ be projective varieties of dimensions r and s respectively. Then every irreducible component of $Y \cap Z$ has dimension $\geqslant r + s - n$. Moreover, if $r + s - n \geqslant 0$ then $Y \cap Z$ is non-empty.*

Idea of proof. The first statement follows by considering the affine patches of the variety. For the second you consider the "cones" over the two varieties in $\mathbb{A}^{n+1}$ and use the Affine Dimension theorem to show their intersection contains a point, and hence so does the original intersection. $\square$

The Hilbert Polynomial.

The idea of The Hilbert Polynomial is to assign to each projective variety $Y \subset \mathbb{P}^n$ a polynomial $P_Y \in \mathbb{Q}[z]$ from which we can obtain various numerical invariants of Y. It involves a whole lot of algebra. First we need:

Def$^{\mathbf{n}}$. A *numerical polynomial* is a polynomial $P \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all sufficiently large $n \in \mathbb{Z}$.

Example $P(z) = \frac{1}{2}z^2 - \frac{1}{2}z$ is a numerical polynomial since $P(n) = n(n-1)/2$ and for any integer n either n or $n - 1$ is even.

Def$^{\mathbf{n}}$. The binomial coefficient function is

$$\binom{z}{r} = \frac{1}{r!} z(z-1)\cdots(z-r+1)$$

for any $z \in \mathbb{R}$ and $r \in \mathbb{N} \cup \{0\}$. It is a polynomial in z of degree r.

Proposition 1. *If $P \in \mathbb{Q}[z]$ is a numerical polynomial of degree r then there exist integers $c_0, \ldots, c_r$ such that*

$$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \ldots + c_r.$$

Proof. By induction on r. $\square$